ENOR: Model Order Reduction of RLC Circuits Using Nodal Equations for Efficient Factorization

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Abstract

ENOR is an innovative way to produce provably-passive, reciprocal, and compact representations of RLC circuits. Beginning with the nodal equations, ENOR formulates recurrence relations for the moments that involve factorizing a symmetric, positive definite matrix; this contrasts with other RLC order reduction algorithms that require expensive LU factorization. It handles floating capacitors, inductor loops, and resistor links in a uniform way. It distinguishes between active and passive ports, does Gram-Schmidt orthogonalization on the fly, controls error in the time-domain. ENOR is a superbly simple, flexible, and well-conditioned algorithm for lightning reduction of mega-sized RLC trees, meshes, and coupled interconnects-all with excellent accuracy.

Introduction

The problem we pose is whether reduction of RLC circuits can be sped up by exploiting the symmetry of such circuits. For example, it has long been known that RLC circuits satisfy reciprocity, which is a sort of symmetry; yet methods like PRIMA [1], when applied to RLC circuits, abrogate this reciprocity. Similarly, the equations for RLC circuits can be formulated so all matrices are symmetric; why not use this nodal form? Ultimately, the speed question leads to the question how to quickly solve large linear systems. Can we formulate RLC reduction so that we factorize only sparse positive definite (spd) matrices?

Motivated by this quest for a more symmetrical computation, **Efficient Nodal Order Reduction**, or ENOR, is a new, provably passive, well-conditioned, reciprocity-preserving, multiport RLC circuit reduction algorithm. It is easy to code, quick to solve, arbitrarily accurate. Possibly, ENOR is the best algorithm to date for reducing colossal RC and RLC circuits.

The method draws on the past decade's rich progress in reduction techniques. It adapts moment matching [2], block techniques [3], Arnoldi-like orthogonalization [4], and congruence transformations [5][1]--ever with an eye to symmetry and its consequent efficiencies. In [6][7] symmetry is used to advantage in RC circuits; ENOR does like service for RLC circuits.

We will derive and explain the method and then illustrate it with several examples.

1 Nodal Equations

Most discussions of model order reduction begin with the modified nodal analysis (MNA) equations for an RLC circuit,

$$\left(\begin{bmatrix} C & 0 \\ 0 & L \end{bmatrix}\frac{d}{dt}+\begin{bmatrix} G & -E \\ E^T & 0 \end{bmatrix}\right)\begin{bmatrix} x \\ i \end{bmatrix}=\begin{bmatrix} B \\ 0 \end{bmatrix} j \qquad (1.1)$$

The unknowns $x \in \mathcal{L}^n$ and $i \in \mathcal{L}^p$ are the nodal voltages and inductor currents, respectively ($\mathcal{L}$ denotes the space of real-valued functions of time defined for t>=0). Blocks $C \in R^{nxn}$ and $G \in R^{nxn}$ are the nodal capacitance and conductance matrices (both are symmetric and positive semi-definite); $L \in R^{pxp}$ is the matrix of self and mutual inductances (L is also symmetric and positive semi-definite); $E \in R^{nxp}$ is the incidence matrix of the inductors. Finally, $j \in \mathcal{L}^N$ is a vector of current sources with incidence matrix $B \in R^{nxN}$.

Advantages of MNA include it's simplicity and, especially, its generality: it can handle voltage sources, for example, and all four types of controlled sources--something plain nodal analysis cannot do without some contriving. However, for RLC reduction we don't need this generality. Besides, (1.1) has the drawback that the coefficient matrices are not at once symmetric and positive-definite, which means pivoting ought to be used during factorization.

For our purposes the nodal equations, obtained by taking the Laplace transform of (1.1) and eliminating inductor currents, are preferable:

$$(Cs + G + \Gamma\frac{1}{s})X(s) = BJ(s) \qquad (1.2)$$

Here $\Gamma = EL^{-1}E^T$ and X and J are the Laplace transforms of x and j.[1] Γ, like C and G, is symmetric and positive

[1] Arguably there is an iota of advantage in working with L^{-1} instead of L. Some 2-D inductance programs, using an analogy

semidefinite; this is an important advantage of (1.2) over (1.1).

We supplement (1.2) by equations to extract the nodal voltages of interest. Conceptually, we partition the nodes of the circuit into three classes: 1) *internal nodes* (those whose identity may be lost during reduction), 2) *active ports* (those nodes to which external circuitry may be attached), and 3) *observation ports* (nodes whose voltages we want to observe but to which no external circuitry may be connected). Denoting active and observation port voltages by u_a and u_o, we can write

$$\begin{aligned} u_a &= B^T v \\ u_o &= B_o^T v \end{aligned} \tag{1.3}$$

where B, the incidence matrix of the active ports, is the same as in (1.2) and B_o is the incidence matrix of the observation ports.

2 Orthogonal Projection and Passivity

An effective way of reducing a system of equations is by projecting them into a suitable subspace.

Definition 2.1. A **orthogonal projection** of equations (1.2) and (1.3) by **projection method** P_V, where $V \in R^{nxm}$, is the m-dimensional system

$$\begin{aligned} &\left(\tilde{C}s + \tilde{G} + \frac{\tilde{\Gamma}}{s}\right)\tilde{X}(x) = \tilde{B}J(s) \\ &U_a(s) = \tilde{B}^T \tilde{X}(s) \\ &U_o(s) = \tilde{B}_o^T \tilde{X}(s) \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} &\tilde{C} = V^T C V \quad \tilde{G} = V^T G V \quad \tilde{\Gamma} = V^T \Gamma V \\ &\tilde{B} = V^T B \quad \tilde{B}_o = V^T B_o \end{aligned} \tag{2.2}$$

m is the **order** of the reduction, and P_V is **proper** if m<n and rank(V)=m.

Orthogonal projection is simply the **Galerkin process**, which historically has been used to replace infinite-dimensional systems (i.e. partial differential and integral equations) by finite-dimensional sets of equations.

We saw that C, G and Γ are positive semidefinite for an RLC circuit. This is important because of the following theorem, proved in the appendix:

Theorem 2.1. Projected system (2.1) is passive if C, G and Γ in the original system are positive semidefinite.

A passive circuit cannot generate energy, is stable, and can be connected to other passive circuits without risk of instability. Orthogonal projection preserves passivity by virtue of its symmetry; oblique projection ($W^T CV$, etc. with $W^1 V$) does not guarantee passivity.

3 Moment Matching

The fidelity of a projected system depends crucially on the choice of the projection matrix V. Most reduction techniques choose V to span the first q moments of x in (1.1), for some q; our contribution lies in applying the idea of moment matching to (1.2), which has greater symmetry, rather than to (1.1).

Let's see how this can be done. It is convenient to change variable from s, the complex frequency, to z,

$$s = s_o(1 - z) \tag{3.1}$$

and then expand $X(s)$ in powers of z; this amounts to expanding about the point s_o and scaling frequencies by $(-1/s_o)$, since by (3.1)

$$z = \frac{-1}{s_o}(s - s_o) \tag{3.2}$$

We also introduce an auxiliary quantity

$$Y(z) = \frac{1}{1 - z} X(z) \tag{3.3}$$

Proceeding as in the method of moments, we expand $X(z)$, $Y(z)$ and $J(z)$ in powers of z, and substitute the expansions into (1.2) and (3.3):

$$\begin{aligned} &\left\{Cs_o(1-z)+G\right\}\left\{X_o + X_1 z + \Lambda\right\} + \frac{\Gamma}{s_o}\left\{Y_o + Y_1 z + \Lambda\right\} = B\left\{J_o + J_1 z + \Lambda\right\} \\ &\left\{X_o + X_1 z + \Lambda\right\} = (1 - z)\left\{Y_o + Y_1 z + \Lambda\right\} \end{aligned} \tag{3.4}$$

Equating powers of z results in the following system of recurrence relations,

$$\begin{aligned} &\left(Cs_o + G + \frac{\Gamma}{s_o}\right)X_k = Cs_o X_{k-1} - \frac{\Gamma}{s_o} Y_{k-1} + BJ_k \\ &Y_k = X_k + Y_{k-1}, \qquad X_{-1} = Y_{-1} = 0 \end{aligned} \tag{3.5}$$

which can be solved for as many terms of the sequence $\{X_k\}$ as desired. The matrices $X_k \hat{I} R^{nxN}$ are, except for the mutliplicative constant $\left(-1/s_o\right)^k$, the (block) **moments** of the nodal voltages when expanded about the frequency s_o.

Instead of moment matching, we could apply projective convolution to (1.2); this would lead to a different set of recurrence relations; see [8].

In solving (3.5), we take $J(z)=I_N$, the NxN identity matrix; this amounts to applying an impulsive current at each active port in turn. The general case, if needed (it isn't needed here), can be found by superposition.

Reduction is achieved by setting $V=[X_o,...X_q]$, i.e. the matrix composed of the first q block moments, and then

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between capacitive charge and inductive flux, calculate L^{-1} directly and then invert. Also, L^{-1} usually is sparser, or at least more diagonally dominant, than L

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Algorithm ENOR
Input: C,G,Γ,B,B_o, and s_o.
Output: V.
V=R^{nx0} initially.
Set X,Y∈R^{nxN} to zero.
Cholesky factorize A=Cs_o+G+Γ/s_o.
RHS=B;
while (not converged)
    Solve AX=RHS;
    Y=Y+X;
    for j=1,2,...,N
        bDeflate=orthnormalize(X(j),Y(j),V);
        if bDeflate
            delete jth column from X and Y
        else
            appendColumn(V,X(j));
    RHS=Cs_oX - ΓY/s_o;

subroutine orthnormalize(x,y,V)
    for each column w of V...
        x=x-(w,x)w;
        y=y-(w,x)w;
    if ||x||<tol
        //deflate;
        return true
    else
        x=x/||x||
        y=y/||x||
        return false
```

Figure 1. The ENOR algorithm

orthogonally projecting (1.2) and (1.3) with this V. If each X_k has N_a columns, the order of V is $(q+1)N_a$.

In practice it is *col-span{[X_o,...X_q]}* and not the identity of the individual X_k*'s* that is important; therefore, to ensure that V is well-conditioned, we orthnormalize the X_k's on the fly, doing deflation as needed and keeping Y_k in synch by applying identical operations to Y_k and X_k during orthogonalization. It can be shown that this Arnoldi-like procedure leaves *col-span{[X_o,...X_q]}* in tact.

There is the practical issue of how many moments to use. We get adaptive error control if we include moments in V until two successive projected systems have the same time-domain responses to within a given tolerance; this requires simulating a number of reduced systems, but the time-step can be quite large and the simulation time short.

The ENOR algorithm is summarized in Figure 1, where subroutine orthnormalize(...) is also detailed.

4 Reciprocity

Besides preserving passivity and matching moments, ENOR has another property that is a consequence of the symmetry of (1.2), namely,

Theorem 4.1. The reduced system (2.1) obtained by ENOR is reciprocal among the active ports.

Proof: A system is reciprocal if its impedance matrix is symmetric. But Z(s) for the active ports of the projected system is

$$Z(s) \equiv \frac{U_a(s)}{J(s)} = \tilde{B}^T (\tilde{C}s + \tilde{G} + \tilde{\Gamma}/s)^{-1} \tilde{B} \qquad (4.1)$$

which is symmetric since $\tilde{C}, \tilde{G}, \tilde{\Gamma}$ are symmetric.

5 Efficient Factorization

It is a question whether symmetric sparse matrix techniques can be as efficient as special-purpose codes like RICE, which reduces simple structures like trees and ladders before solving several smaller systems by general techniques[9]. From a software perspective, RICE has the disadvantage of needing to handle floating capacitors, inductor loops, and resistor links as separate cases; ENOR handles all of these 'cases' in a uniform way.

As indicated in Figure 1, ENOR must factor the symmetric positive semidefinite (*sps*) system

$$A = Cs_o + G + \frac{\Gamma}{s_o} \qquad (5.1)$$

In practice, *A* in (5.1) is almost always positive definite (rather than just *semi*definite). The following theorem provides sufficient conditions for this:

Theorem 5.1. Matrix *A* in (5.1) will be positive definite if each node in the corresponding RLC circuit has some path through resistors and capacitors to ground.

Proof: Arguing physically, we show that $x^T(Cs_o + G)x > 0$; then, certainly, $x^T Ax > 0$, since Γ is sps. Interpret Cs_o+G as a resistive network obtained from the RLC circuit by retaining resistors, dropping inductors, and replacing each capacitor *c* by a conductance cs_o. As *x* is the vector of nodal voltages, $x^T(Cs_o+G)x$ is the power dissipated in the resistive network. Unless all nodes are grounded ($x=0$), there will be some dissipation, or $x^T(Cs_o+G)x>0$, because each node has a resistive path to ground.

Because of Theorem 5.1, we can, in practice, Cholesky factorize *A*: pivoting is not needed, and we can order nodes before-hand to reduce fill. For large RLC networks, this is a significant benefit.

We believe that sparse Cholesky factorization, if implemented properly, can be as efficient as a special-structure handling method like RICE. In the case of trees, for example, Partner has shown that if a **monotone**

ordering is used (children are numbered before their parents), there is zero fill when factorizing trees [10]. ENOR is *O(n)* for trees and its performance for more general topologies, including meshes and coupled trees, is also excellent.

6 Examples

We next give several examples of the efficacy of ENOR in reducing RLC circuits. In all examples, we take $s_o=1x10^9$ Hz.

6.1 Coupled Lines

Consider a pair of 4 inch lines modeled by 40 coupled RLC sections (R=0.014 ohm, L=1 nH, C=0.4 pF, and C_m=0.08 pF per section). With all lines terminated in 50 ohms, Figure 2 plots the response at the four ports of the original (order 162) and reduced (order 16) systems; the responses are indistinguishable, max error being $8x10^{-5}$ V.

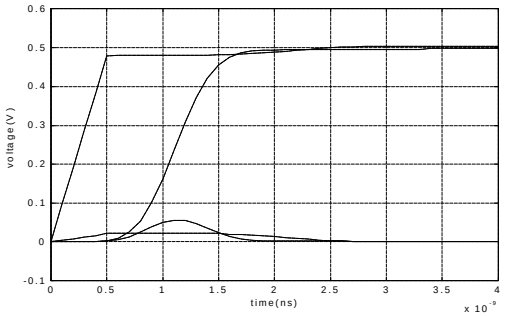

Figure 2. Crosstalk between a pair of lines

6.2 Balanced Clock Tree

Next consider a 5-level balanced tree such as might be encountered in routing a clock; each branch is a 2 inch, 50 ohm line, with a 1 pF load at each leaf. The root, which is treated as a active port, has a 50 ohm, 0.5 ns linear driver. All leaves and internal nodes are set to observation ports.

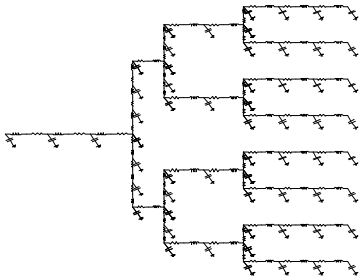

Figure 3. Balanced clock tree

When the 621 node tree is reduced to order 13, the responses before and after reduction, shown in Figure 4 for each observation port, agree within $1.5x10^{-4}$ V.

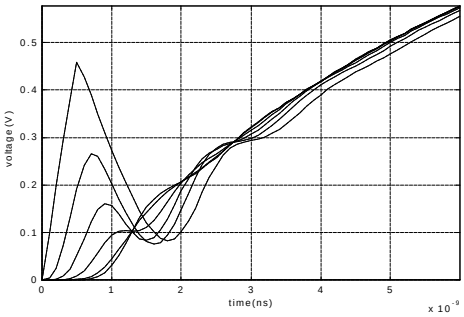

Figure 4. Response of clock tree

6.3 Transmission Line Mesh

Finally, consider an 5x5 mesh of 1 inch, 50 ohm transmission lines which, modeled by 3 RLC sections per inch, is a 225 node circuit:

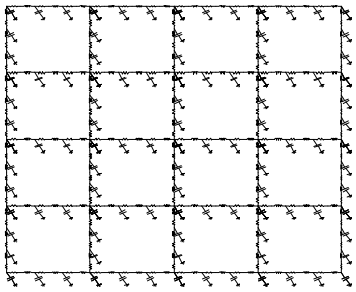

Figure 5. 5 x 5 mesh

The lower, left corner of the mesh is an active port; all other junctions of the mesh (14 in all, accounting for symmetry) are observation ports. As before, a 50 ohm, 0.5ns ramp drives the active port. The original and an order 20 reduced circuit are compared in Figure 6 (max error $1.5x10^{-4}$).

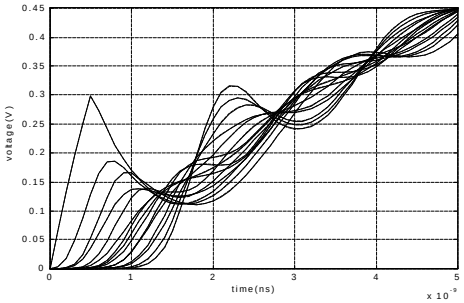

Figure 6. Response of mesh (Na=1)

How does the order change if we use more active ports? If the 5x5 mesh is modeled with 3 active ports (lower-left, center, and upper-right), it turns out that an order 27 system is needed to achieve the same accuracy. With one active port, (3.5) is solved for q=20 moments, each having Na=1 columns. With 3 active ports, (3.5) is solved for q=9

moments, each having Na=3 columns. In general, the more active ports, the fewer block moments required; for a given accuracy, however, the net order qN_a increases with N_a.

Conclusion

In this paper we have presented ENOR, a new order-reduction method for large RLC circuits that produces both passive and reciprocal macromodels. ENOR distinguishes between active and observation ports to allow maximum compression. It does moment matching like other methods, but works with the nodal equations, which are more symmetric than MNA. The ENOR formulation leads to a symmetric positive-definite matrix, so sparse Cholesky techniques--which are, we think, competitive with path-tracing--apply. Also, it does Gram-Schmidt orthogonalization on moments as they are computed (a process that is arguably simpler than the block Arnoldi methods in the literature). It controls error adaptively, in the time-domain. Powered by these innovations, ENOR renders excellent accuracy and compression ratios for coupled traces, trees, and meshes--all with inductance.

Appendix

Theorem 2.1. Projected system (2.1) is passive if C, G and Γ in the original system are positive semidefinite.

Proof: To be passive, transfer function (4.1) must satisfy the following conditions [11]:
(1) Z(s) is a rational function of s.
(2) $Z(s^*) = [Z^T(s)]^*$ for all complex s, where * means conjugate transpose, and
(3) $z^*(Z(s)+Z^*(s))z \geq 0$ for all complex vectors z and complex s with Re(s)>0.
Write T=Cs+G+Γ/s. First, Z(s) is rational because the inverse of a rational matrix (i.e. T) is rational. Secondly, C, G, Γ and B being real, conjugating Z(s) merely conjugates s. Condition (3) requires

$$z^* B^T (T^{-1} + T^{-*}) Bz \geq 0$$

where $T^{-*} \equiv [T^{-1}]^*$. But this is the same as

$$z^* B^T T^{-*} (T^* + T) T^{-1} Bz =$$
$$w^* (s(C + C^T) + (G + G^T) + \frac{s}{|s|^2}(\Gamma + \Gamma^T))w \qquad (2.2)$$

with $w = T^{-1}Bz$ and $s = s + jw$; in arriving at (2.2), we have used the fact that $w^*(C - C^T)w$ and $w^*(\Gamma - \Gamma^T)w$ are both zero, since the transpose of a scalar equals itself. Since C, G and Γare positive semi-definite, (2.2) is greater or equal to zero for all *w*, hence for all *z*, as required.

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