

Discharge Current Steering for Battery Lifetime Optimization

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ABSTRACT

Recent work on battery-driven power management has demonstrated that sequential discharge is suboptimal in multi-battery systems, and lifetime can be maximized by distributing (steering) the current load on the available batteries, thereby discharging them in a partially concurrent fashion. Based on these observations, we formulate multi-battery lifetime maximization as a continuous, constrained optimization problem, which can be efficiently solved by non-linear optimizers. We show that great lifetime extensions can be obtained with respect to standard sequential discharge, as well to previously proposed battery allocation schemes.

Categories and Subject Descriptors

J.6 [Computer Applications]: Computer-Aided Engineering; C.4 [Computer Systems Organization]: Performance of Systems; G.1 [Numerical Analysis]: Optimization

General Terms

Design, Performance

Keywords

Energy consumption, battery lifetime optimization

1. INTRODUCTION

Supporting multi-battery power supplies is becoming standard for modern electronic products, as this option enables the user to trade battery lifetime for weight upon needs. From the manufacturing standpoint, there is clearly a number of issues that must be faced when multiple batteries have to be accommodated into the case of an electronic product, such as a laptop or a cell phone. They range from the selection of battery capacities and shapes, to the design of the power supply circuitry (including the switching regulator that interfaces the various batteries to the current load).

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ISLPED'02, August 12-14, 2002, Monterey, California, USA.
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One degree of freedom that, so far, has not been fully exploited, is the policy to be used for discharging the available batteries. In other words, the approach adopted in existing products consists of fixing, once and for all during system design, the order in which batteries have to be discharged. A battery is not disconnected from the current load until it is exhausted.

The rationale for this solution stands in the assumption that batteries well approximate ideal charge storage. In other words, the amount of charge (i.e., the capacity) a battery can deliver is independent of the way the charge is extracted. Unfortunately, the behavior of a real battery is far different from the ideal case. In particular, at higher current loads, a battery is less efficient in converting its chemically stored energy into available electrical energy; thus, its actual capacity deviates more sensibly from the nominal value. This effect has been discussed in a number of previous works [1, 2, 3, 4]. Accurate battery modeling is a complex task, because a number of additional non idealities needs to be taken into account (e.g., battery recovery, internal resistance, thermal effects). However, load-dependent capacity is the most significant non-ideality in real-life batteries and all battery manufacturers provide quantitative data on this effect in their data-sheets (discharge curves, plotting capacity vs. current load), while other, more complex effects, such as charge recovery require additional, extensive experimental characterization.

In multi-battery systems, the load-dependent capacity of batteries has profound implications. First and foremost, the commonly accepted sequential discharge schedule is a very inefficient policy from a battery lifetime viewpoint, as observed in [5, 6]. Pedram, Wu and Qiu [5] propose an improved battery discharge policy that selects the battery to be connected to the load based on the absorbed current level. This strategy is effective when batteries are highly asymmetric (e.g. one battery is very efficient at low currents, while the other is much better at high currents), and when the current load has high variance. Unfortunately, load-based battery switching degenerates to sequential discharge for multiple cells that respond similarly to the load (e.g., multiple equal batteries, similar batteries with different size) and for constant loads.

An alternative approach to multi-battery scheduling was proposed in [6]. One of the main results obtained in that work was that, given a simple two-battery system made of two identical battery cells, significant lifetime improvements can be achieved by alternatively connecting the two battery cells to the load. The higher the frequency at which bat-

teries are switched, the closer the lifetime gets to that of a monolithic battery having double capacity. This result can be intuitively explained by the fact that, as the switching frequency increases, the current load tends to “see” the two batteries as if they were connected in parallel (the parallel connection of voltage sources is obviously not allowed). Thus, the current load appears as equally split over the two batteries, which are operated under less demanding load, and therefore deliver a larger amount of charge. In some specific applications (for instance, medical devices) high switching frequencies cannot be tolerated. Fortunately, in these cases, the ability of a battery to recover some of its deliverable charge if periods of discharge are interleaved with rest periods can be exploited to achieve sizable lifetime improvements of the multi-battery system [7].

In this paper, we build upon the idea of fast alternation of battery usage by generalizing the concept of current splitting introduced in [6]. We consider multi-battery power supplies containing heterogeneous cells, that is, cells having different nominal capacities and discharge curves. For this kind of power supplies, it is clear that the amount of current that should be drawn from each cell, when the “parallel” battery is created through fast switching, must vary according to the actual capacity of each cell. In other words, the current load should be split non-uniformly over all the cells in the power supply. This objective is achieved by a fast-switching round-robin policy that connects battery cells to the load for time periods of different duration. For example, in the case of a two-battery system, this translates to a connection of the cells to the load following a square wave with unbalanced duty-cycle.

To optimally solve the problem of choosing the right discharge current for each battery (or, equivalently, to optimize the duty cycle of the switch control waveform), we cast it as a continuous, constrained optimization problem and we solve it using standard optimization methods. The assumption on which our formulation is based is that the profile of the current load to be drawn from the power supply is well characterized, that is, the percentage of the time in which the system operates at different current levels is known.

We present experimental results showing that the new current allocation method yields lifetime extensions over previously published allocation algorithms [6] as high as 12%; lifetime extensions with respect to sequential battery discharge are clearly much larger than this, as they can be as high as 160%

2. PROBLEM FORMULATION

2.1 Capacity vs. Lifetime

Because of the non-idealities of real batteries, the relation between discharge time T (the *lifetime*, hereafter) and battery capacity C cannot be simply derived by the ideal battery capacity formula $C = T \cdot I$, where I is the discharge current.

This fact is at the basis of the well-known Peukert’s equation [8], that models non-idealities for the case of a constant current load by introducing a penalty value that decreases the actual capacity (i.e., the battery efficiency) for larger current loads. Peukert’s equation relates C and T as follows:

$$C = T \cdot I^\alpha \quad (1)$$

where $\alpha > 1$ is called the Peukert’s value. Typical values of α are between 1.2 and 1.4. The value $\alpha = 1$ represents the ideal case.

For our purposes, we model this non-ideality in a slightly different way, similar to the approach followed by Pedram and Wu in [9]. This alternative solution consists of expressing the dependency between capacity and discharge current as follows:

$$C = T \cdot I \cdot \rho_I \quad (2)$$

In Equation 2, $\rho_I > 1$ is the *current scaling factor*, which accounts for the fact that the battery is less efficient in using its capacity for larger current values. ρ_I actually expresses the ratio of the nominal capacity of the battery C_0 , and the discharge characteristics of the battery versus the load current. In formula, $\rho_I = \frac{C_0}{C(I)}$. Re-arranging the equation for C , we get:

$$T = \frac{C(I)}{C_0 I} \quad (3)$$

2.2 Load Characterization

The constant current load at the basis of Peukert’s equation is not a realistic assumption for real-life systems, that are typically characterized by variable loads. One possibility to model a variable load is to assume a set of M current levels (I_1, \dots, I_M) ; their distribution over time is described by a set (x_1, \dots, x_M) , where x_i denotes the percentage of total operation time spent with current I_i .

Such characterization can be achieved by statistical profiling of the typical behavior of the system under analysis over a significant period of time. The value of M determines the quantization interval used to characterize the load. Although somehow simplistic, this model can be tuned with arbitrarily fine accuracy, and is general enough.

Under this load model, Equation 2 generalizes to:

$$C = T \cdot \sum_{i=1}^M x_i \cdot I_i \cdot \rho_{I_i} \quad (4)$$

Equation 4 is at the basis of our formulation of the load optimization problem described in the next subsection.

2.3 Load Optimization

We assume that N batteries are available, each one characterized by its capacity equation $C_i(I)$, $i = 1, \dots, N$. Our formulation of the load optimization problem is based on the following two assumptions:

- All the N batteries are discharged concurrently, and the load current is partitioned, in general not equally, among them. This is equivalent to assuming round-robin switching policy which connects each battery to the load for a time proportional to the fraction of the load current we wish to absorb from the battery. The cycle frequency of the round-robin schedule is fast enough to let batteries perceive only its time-averaged effect (i.e., a constant current equal to a fraction of the load current).
- Since we are discharging the N batteries concurrently, we wish to fully discharge all of them at the same time T , that represents the lifetime of the overall battery pack. Any differences in time-to-total discharge among batteries can be seen as an inefficiency in the current

steering policy, and we want to eliminate the inefficiency by construction.

The load optimization problem can be formulated as a non-linear optimization problem as follows:

Maximize T , such that:

$$\begin{cases} I_1 = I_{1,1} + \dots + I_{1,N} \\ \dots \\ I_M = I_{M,1} + \dots + I_{M,N} \end{cases} \quad (5)$$

$$\begin{cases} 1 = T \cdot \sum_{i=1}^M \frac{x_i \cdot I_{i,1}}{C_1(I_{i,1})} \\ \dots \\ 1 = T \cdot \sum_{i=1}^M \frac{x_i \cdot I_{i,N}}{C_N(I_{i,N})} \end{cases} \quad (6)$$

subject to:

$$0 \leq I_{i,j} \leq I_i, \quad i = 1, \dots, M \quad j = 1, \dots, N \quad (7)$$

The unknowns (decision variables) are:

- The lifetime T .
- The currents $I_{i,j}$, that define what fraction of the current I_i is to be extracted from battery j . For M current levels and N batteries, there are $N \cdot M$ such currents.

The problem is subject to three types of constraints:

- *M current equality constraints* (Equation 5): These express the fact that all the battery-loading currents $I_{i,j}$, $j = 1, \dots, N$, in each load condition, must sum up to the corresponding total load current I_i must sum up to I_i .
- *N battery equality constraints* (Equation 6): These constraints are obtained by Equation 3, applied to each battery. They express that, for each battery j , $j = 1, \dots, N$, the various current loads $I_{i,j}$ allocated to it must discharge the battery at time T . This condition must hold for all the batteries, which discharge at the same time T .
- *$M \cdot N$ bound constraints* (Equation 7): These simply express the fact that each of the sub-currents on each battery $I_{i,j}$ must be (i) nonnegative quantities, and (ii) must not exceed the corresponding total current I_i . Since the upper bound of this set of constraints is implicitly contained in Equation 5, they can be simplified to $I_{i,j} \geq 0$.

In spite of the simplicity of the objective function, the problem is far from having a trivial solution, because the equality constraints of Equation 6 are in general non-linear. A local minimum can be found using standard continuous non-linear optimizers [10] (e.g., quasi-Newton, gradient). Notice that the size of the problems is not a major concern because the number of load levels and the number of battery cells in the system is unlikely to be very large.

One important observation about the above formulation concerns the battery models. It is important to emphasize that reducing the battery behavior to a capacity equation

only approximately models the complex behavior of real-life batteries; other effects such as charge recovery due to battery idleness are not included in this model. Notice, however, that the proposed allocation scheme is insensitive to the recovery effect, because it is based on a high-frequency alternation of the various battery packs.

The following is a simple instance of the problem that shows how the current allocation is superior, for instance, to a sequential discharge scheme.

EXAMPLE 1. Consider a system with $N = 2$ batteries, and a constant current load of $2A$. This corresponds to the case $M = 1$, with $I_1 = 2$, and $x_1 = 100\%$. For simplicity, let us assume that the two batteries have the following linear capacity equations (in some unit of charge):

- $C_1(I) = 10 - I$
- $C_2(I) = 15 - 2 \cdot I$

The unknowns of the problem are the battery lifetime T , and the two sub-currents $I_{1,1}$ and $I_{1,2}$ that specify which fraction of I_1 is allocated to battery 1 and 2, respectively. The problem formulation is the following:

Maximize T , such that:

$$\begin{cases} 2 = I_{1,1} + I_{1,2} \\ 1 = T \cdot \frac{I_{1,1}}{10 - I_{1,1}} \\ 1 = T \cdot \frac{I_{1,2}}{15 - 2 \cdot I_{1,2}} \end{cases}$$

subject to:

$$\begin{cases} 0 \leq I_{1,1} \leq 2 \\ 0 \leq I_{1,2} \leq 2 \end{cases}$$

The above formulation admits one solution only, since there are three equalities for three unknowns; solving the system yields $T = 10.919$. This optimum corresponds to the values of $I_{1,1}$ and $I_{1,2}$ of $0.839A$ and $1.161A$, respectively. In practice, given the characteristics of the two batteries, the best choice is to allocate $41.95\% = (0.839/2)$ of I_1 to the first battery, and $58.05\% = (1.161/2)$ to the second one.

Let us now compare this value with the sequential discharge of the two batteries. Since the load current is constant, the order of discharge is roughly irrelevant. The first battery, when discharged with $I = 2A$ has an effective capacity $C_1(I) = C_1(2) = 10 - 2 = 8$. Under current I , this effective capacity corresponds to a duration $T_1 = \frac{C_1(I)}{I} = \frac{8}{2} = 4$. Using the same calculations, the second battery has an effective capacity $C_2(I) = C_2(2) = 15 - 2 \cdot 2 = 11$, corresponding to a duration $T_2 = \frac{C_2(I)}{I} = \frac{11}{2} = 5.5$, for a total duration of $T_s = 4 + 5.5 = 9.5$, a 13% shorter battery lifetime than the optimal value.

3. EXPERIMENTAL RESULTS

We have used a standard commercial package to solve the current allocation algorithm, namely, the nonlinear constrained optimizer of the Matlab Optimization Package. It uses the so-called Sequential Quadratic Programming (SQP) methods, consisting of an iterative solution of several quadratic programming sub-problems, that have been

shown to represent the state of the art in nonlinear programming methods [10].

To validate the effectiveness of the proposed solution we have run several experiments to compare the lifetime achievable with *proportional current allocation* (i.e., the method of this paper) with respect to the uniform current splitting approach of [6] (referred to in the sequel as *fixed current allocation*). Comparison to lifetimes provided by sequential battery discharge (which is the policy adopted by existing electronic products) is also provided for the sake of completeness.

3.1 Explorative Analysis

In a first experiment, we have evaluated the impact of proportional current allocation for a number of variants of a reference workload applied to a system. The system consists of two batteries, whose capacity equations are:

- a) $C_1(I) = 10 \cdot (1 - 0.04 \cdot I^{1.4})$
- b) $C_2(I) = 15 \cdot (1 - 0.08 \cdot I^{1.3})$.

These models have been obtained by fitting the discharge profile of two real-life batteries to a generic equation template of the form $C_0 \cdot (1 - \alpha \cdot I^\beta)$.

The workload consists of two currents of $I_1 = 1A$ and $I_2 = 3A$. We have then analyzed the discharge of the two batteries for different distributions of the workload; in particular, according to the formulation of Section 2, we have applied various workloads consisting of I_1 and I_2 and different values of x_1 and x_2 . Figure 1 plots, for different values of the ratio x_1/x_2 , the lifetime of the battery system obtained by solving the proportional current allocation problem (*Proportional*) versus the lifetime corresponding to fixed current allocation (*Fixed*).

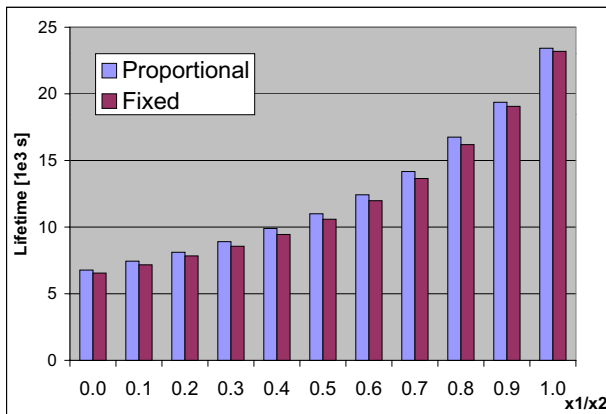


Figure 1: Comparing Lifetimes of Proportional and Fixed Current Allocation.

The plot shows that proportional current allocation roughly provides a fixed amount of lifetime extension, that has a more significant impact on heavier workloads ($x_1/x_2 \leq 0.5$), signifying that in case of higher currents there is higher margin for current allocation than for smaller currents. This fact is also shown in Figure 2, where the percentage of lifetime extension is plotted for three different workloads: The first one consists of the same current levels as in Figure 1

($I_1 = 1, I_2 = 3$), whereas the second and the third have ($I_1 = 2, I_2 = 4$), and ($I_1 = 4, I_2 = 8$), respectively. In the plot we notice how the lifetime extensions increase as the average value of the current drawn increases, and are also more sensitive to the ratio x_1/x_2 .

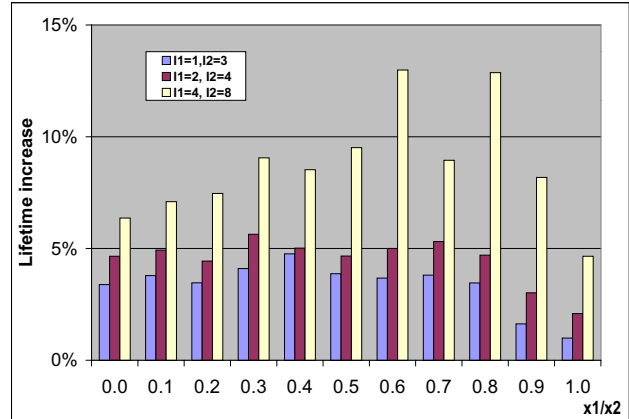


Figure 2: Lifetime Extensions using Proportional Current Allocation for Different Current Levels.

3.2 Synthetic Workloads

Another type of validation has been carried out over a set of synthetic, i.e., artificially generated workloads, characterized by different current levels and time-domain behaviors. More specifically, we have considered a total of 6 types of current load stimuli, characterized as follows:

- Type *CC*: 2 constant current loads of magnitude 0.1 and 1.0A.
- Type *SSW*: 2 symmetric square waves (50% duty-cycle), with average value of 0.5A, and different current levels: (0.4A, 0.6A) and (0.2A, 0.8A).
- Type *ASW*: 2 asymmetric square waves (20-80% duty-cycle), with average value of 0.5A, and different current levels: (0.4A, 0.6A) and (0.2A, 0.8A).

These workloads have been applied to the case of two, three and four batteries, whose capacity vs. current equations are shown in the following table (in *mAh*):

B_1	$C_1(I) = 1000 \cdot (1 - 4 \cdot 10^{-5} \cdot I^{1.4})$
B_2	$C_2(I) = 1500 \cdot (1 - 2 \cdot 10^{-4} \cdot I^{1.3})$
B_3	$C_3(I) = 2000 \cdot (1 - 5 \cdot 10^{-6} \cdot I^2)$
B_4	$C_4(I) = 1000 \cdot (1 - 1 \cdot 10^{-4} \cdot I^{1.2})$

The two battery case consists of the combination of B_1 and B_2 , while the three-battery case consists of batteries B_1, B_2 and B_3 .

Tables 1- 3 compare the lifetime T achieved by proportional current allocation (Column *Prop*) to that achieved through fixed current allocation (Column *Fixed*), as well as to that given by sequential battery discharge (Column *Seq*), for the various workloads and for the three battery configurations.

Workload	Lifetime [s]				
	Prop	Fixed	Δ [%]	Seq	Δ [%]
CC1	88886	88652	0.26	87365	1.74
CC2	6676	6126	8.98	3556	87.74
SSW1	16348	15864	3.05	13545	20.69
SSW2	15726	14415	9.09	9617	63.52
ASW1	14248	13818	3.11	11413	24.84
ASW2	9873	9356	5.52	6052	63.14

Table 1: Lifetime Comparison for Synthetic Workloads (2 Batteries)

Workload	Lifetime [s]				
	Prop	Fixed	Δ [%]	Seq	Δ [%]
CC1	161400	161143	0.16	158760	1.66
CC2	14150	12622	12.10	5443	159.96
SSW1	31393	30428	3.17	25175	24.69
SSW2	30489	28465	7.11	16794	81.54
ASW1	27707	26374	5.05	21387	29.55
ASW2	20239	18589	8.88	10517	92.44

Table 2: Lifetime Comparison for Synthetic Workloads (3 Batteries)

Workload	Lifetime [s]				
	Prop	Fixed	Δ [%]	Seq	Δ [%]
CC1	197495	197135	0.18	193860	1.87
CC2	18148	16449	10.33	7608	138.54
SSW1	38981	37771	3.20	31233	24.81
SSW2	38118	36133	5.49	22147	72.11
ASW1	34429	33326	3.31	26565	29.60
ASW2	25675	23657	8.53	13982	83.63

Table 3: Lifetime Comparison for Synthetic Workloads (4 Batteries)

Figure 3 pictorially summarizes (with data grouped by workload type) the percentage lifetime increase provided by proportional current allocation over fixed current allocation, as this represents the closer target of the method introduced in this paper. The increase tends to be larger for workloads in which the asymmetry in the battery can be fully exploited (i.e., higher current levels, and larger variance in the levels). For instance, workloads of type *SW2* or *ASW2* yield better solutions than *SW1* and *ASW1*, respectively.

3.3 Real-Life Example

The third experiment we have carried out consists of evaluating a real-life workload extracted from an actual system, and perform some exploration about the opportunities offered by various battery configurations.

The system is a digital audio recorder described in [6], that can operate in four active states with sensibly different current absorptions, summarized in the following table:

State	Current [mA]
Sleep	$I_1 = 15$
Idle	$I_2 = 220$
RawSound	$I_3 = 460$
FineSound	$I_4 = 760$

A typical usage of the system consists of an alternate, aperiodic sequence of active (playing sound) and idle (silence)

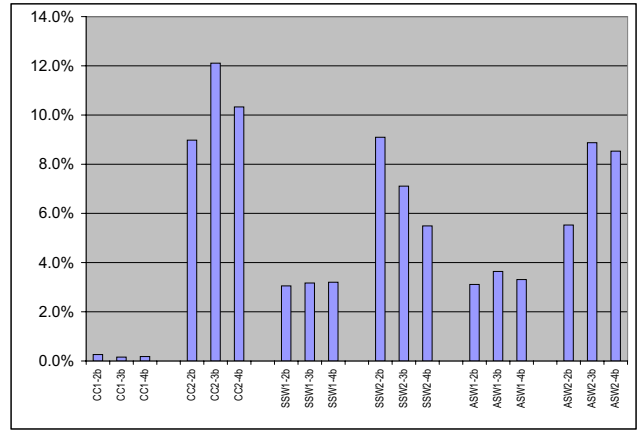


Figure 3: Lifetime Increase Grouped by Workload.

intervals. We have taken a sample usage trace of the system over a significant amount of time, and we translated into our abstract representation of a workload, namely, a set of pairs (*current level*, *percentage of time*). The resulting current profile is $[(I_1, 17\%), (I_2, 14\%), (I_3, 24\%), (I_4, 45\%)]$. We have tested this workload against three different battery system configurations; batteries have been picked from the four packs described in the previous section. The details of the various configurations we consider are the following:

- *BS1*: Two instances of battery B_3 .
- *BS2*: One instance of battery B_3 , and one instance of battery B_4 .
- *BS3*: One instance of battery B_3 , and one instance of a small, backup battery, whose capacity vs. current equation $C_b(I) = 1200 \cdot (1 - 7e - 5 \cdot I^{1.4})$ is totally dominated by that of B_3 .

The following table again compares the lifetime of the battery system obtained by using current steering to the one obtained with fixed current allocation.

Battery Config	Lifetime [s]		Δ [%]
	Prop	Fixed	
<i>BS1</i>	29419	29419	+0
<i>BS2</i>	20617	19118	+7.8
<i>BS3</i>	21728	19194	+13.2

Table 4: Lifetime Comparison for Real-Life Workloads.

The results confirm the trend exhibited by the previous experiments. In particular, for the *BS3* configuration, the fixed allocation scheme is significantly sub-optimal with respect to a careful assignment of the currents.

Notice that for the case of identical batteries (*BS1*), the fixed and the proportional allocation correctly provide the same result. The solution returned by the current allocation algorithm correctly splits each current level equally across the two batteries.

4. CONCLUSIONS

Battery management has shown to be a promising approach to extend lifetime of portable electronic appliances. This is particularly true when the devices are equipped with multi-battery power supplies.

In this paper, we have formulated and solved the problem of optimally allocating current loads to the various cells of a multi-battery system in order to achieve battery lifetime maximization.

The proposed solution consistently outperforms the results given by a current allocation policy that equally partitions the current load to all the batteries available in the power supply. Obviously, the new policy is also greatly superior to sequential battery discharge, the latter being the battery discharge policy adopted by modern electronic products.

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