A Polynomial Upper Bound for a Mutation-Based Algorithm on the Two-Dimensional Ising Model

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Abstract. Fitness functions based on the Ising model are suited excellently for studying the adaption capabilities of randomised search heuristics. The one-dimensional Ising model was considered a hard problem for mutation-based algorithms, and the two-dimensional Ising model was even thought to amplify the difficulties. While in one dimension the Ising model does not have any local optima, in two dimensions it does. Here we prove that a simple search heuristic, the Metropolis algorithm, optimises on average the two-dimensional Ising model in polynomial time.

Keywords: Ising model, expected optimisation time, adaption

1 Introduction

In 1925, Ising [5] developed the Ising model in order to study the effects of ferromagnetism. Naudts and Naudts [7] introduced a class of interesting fitness functions based on this model to the genetic algorithm community. In its most general form the fitness function is defined on colourings of the vertices of weighted, undirected graphs. Given an undirected graph $G = (V, E)$ and a weight function $w : E \mapsto \mathbb{R}$, the fitness of a colouring is defined as the sum of the weights of all monochromatic edges, i. e. edges connecting vertices of equal colour. This function is to be maximised. In its general form, the Ising model is NP-hard, though there exist quite efficient algorithms for this problem (see for example [8,6,2]).

In this article we use the constant weight function 1 for all edges and two colours 0 and 1, which yields a trivial optimisation problem, but an interesting fitness function for the investigation of the adaption capabilities of randomised search heuristics. While nowadays GAs and search heuristics are mainly used as optimisers (see e. g. [3]), they were designed as adaption systems by Holland [4].

In general, most of the graphs are of no interest as regards physics. Here, we focus on a two-dimensional Ising model, or torus, which is – from a physicist's point of view – the most relevant one. Usually, a torus is defined as a set of vertices placed on a $X \times Y$-lattice with edges connecting horizontal and vertical neighbours plus the wrap-around edges connecting vertices in the last column

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(row) with the vertex in the first column (row) of the same row (column), respectively. Here we also use diagonal edges, which is natural since the Ising model was designed as a model for the behaviour of interacting particles, where the strength of the interaction decreases with distance. Throughout this paper, we will restrict ourselves to quadratic toruses, i. e. with n vertices and $X = Y = \sqrt{n}$.

GA experiments on the one-dimensional Ising model and similar fitness functions have been reported by Van Hoyweghen [9] stating that "the Ising model can be seen as an archetypical problem where symmetry prevents an unspecialized GA from solving the problem quickly." Similar statements can be found in [10] and [11].

We aim to analyse a randomised search heuristic on the torus. It is easy to see that there exist non-optimal colourings from which a hill climber can hardly escape. Imagine a torus coloured 1 in the upper half and 0 in the lower half. Flipping only a few bits, the fitness can only decrease, since the flipping bits must produce islands or dents, introducing new non-monochromatic edges. In this situation hill climbers like the (1+1)EA have exponential waiting time. The simplest non-hill climbing selection strategy is used by the Metropolis algorithm:

1. Among all colourings of V, choose x uniformly at random.
2. Repeat
   a) Create x' from x by flipping the colour of a single vertex selected uniformly at random.
   b) Replace x by x' with probability $e^{\min\{0,(f(x')-f(x))/T\}}$.

For $T \to 0$ the algorithm never accepts offspring with lesser fitness and becomes randomised local search (RLS). We apply a local search operator, which is common for the Metropolis algorithm. Using a global search operator that flips each bit with probability $1/n$ independently, as is usually done for evolutionary algorithms, would also be possible. While the analysis gives little further insight, using global mutation inflates a tedious case differentiation to several pages.

A step flipping exactly one bit can change the fitness by at most 8, since each vertex has exactly 8 neighbours. If we set the parameter T to $-2/\ln p_T$ for some $p_T \in [0,1]$, the probability of accepting a step decreasing the fitness by i is at most $p_T^{i/2}$. We will use this fact to limit the number of fitness-decreasing steps.

Section 2 introduces characteristic properties of colourings which decide whether or not it is easy to improve the fitness. Section 3 and 4 deal with two classes of colourings which do not require acceptance of individuals with lesser fitness. Section 5 will describe how the algorithm can escape from the local optima we already mentioned. We finish with some conclusions.

Note that in all images in this article vertices are represented by squares; the implicit edges are omitted. Gray indicates colour 1, white indicates colour 0.

2 Structure of the Torus and Sketch of Analysis

The optimisation process can be seen as a struggle between the two colours. A connected area can dominate another area, if it surrounds it completely. We will show that islands being surrounded by vertices of the other colour shrink and finally vanish. As this happens, the fitness increases from time to time. At their outermost borders, islands are delimited by blocks which can grow and shrink, where shrinking is at least as probable as growing. We can describe this process as a random walk of the block length. When there are none of these blocks or dents left, the colouring consists of rings winding around the torus once or more, and there is no dominating colour. Still the size of a ring may vary and perform a random walk, unless all of the borders of the ring are exactly parallel to edges. This is the only situation in which fitness decreasing steps are necessary. We will now give formal definitions of these notions. If we place the vertices of the torus in a lattice, we can talk about neighboured vertices using the terms "left", "right", "above", and "below". With respect to a directed path p, we can also talk of neighbours of path vertices as being situated left or right of the path.

Definition 1 (Border). *Let $G = (V, E)$ be a torus and $C \subseteq V$ a maximal connected set of vertices coloured c. A path p is called* border *of C if all vertices of p lie in C and all vertices left of p do not lie in C and p cannot be extended without violating the first two conditions.*

With respect to the lattice all edges in p can be assigned a vector from $\{-1, 0, 1\}^2$ specifying its direction in the lattice. The path p is called x-monotone (y-monotone) if for all edges in p the x-component (y-component) of the vector associated with the edge is either always non-negative or always non-positive. p is called monotone *if it is x-monotone and y-monotone. Otherwise p is* bent*.*

Since the torus is finite, all borders are loops. Note that monotone borders meet their starting point after making, e. g., $\sqrt{n}$ steps up and running into their starting point from below. This is why we call the areas delimited by monotone borders "rings."

We will now define two promising classes of colourings. The first definition is straightforward and covers the dents or blocks delimiting an island while the second definition covers special cases.

Definition 2 (Delimiting block). *Let $G = (V, E)$ be a torus and $B \subseteq V$ a connected set of vertices situated on a horizontal or vertical line in the lattice. Let N be the set of vertices that are neighbours of B-vertices situated in any fixed half-plane defined by this line. B is called a* delimiting c-block *of* length $|B|$*, iff all vertices in B are coloured c and all vertices in N are coloured $1 - c$.*

Since the outermost vertices in a delimiting block B have at least four neighbours that are coloured complimentarily, these vertices can always flip and hence decrease the size of B. Possibly the block can grow, but the probability of this event is at most as large as the probability for shrinking, if $|B| \geq 2$.

In the first part of the analysis we want to show that bent borders vanish leaving only rings behind. We do so by showing that the fitness increases as long as we have delimiting blocks. Unfortunately, not all colourings with bent borders do also contain delimiting blocks. In a checkerboard-like colouring, borders run counter-clockwisely around the individual fields and are therefore bent, though not containing any delimiting blocks. Despite of this, it is not hard to improve the fitness by flipping vertices near the corners, where borders intersect each other. We will show that for subgraphs of this type fitness improvement is easy.

Definition 3 (Intersection). *In a coloured torus the following subgraphs and their horizontal and vertical mirages as well as rotations are called* intersection*:*

$$\textit{Type 1:}\ \begin{matrix}101\\011\end{matrix} \qquad \textit{Type 2:}\ \begin{matrix}100\\011\end{matrix} \qquad \textit{Type 3:}\ \begin{matrix}101\\010\end{matrix}$$

We will show that in a colouring with bent borders there must exist at least one delimiting block or an intersection. In both cases a fitness improvement is easy.

Since we will frequently encounter random walks, we will start by providing for an important tool heavily used throughout the analysis. Here we consider a Markov chain that is "at least fair."

Lemma 1. *Consider a Markov chain with transition probabilities* $p(i, i+1) \geq 1/2$ *and* $p(i, i-1) = 1 - p(i, i+1) \leq 1/2$*. Starting at state* 0 *the probability of reaching a state* $m \geq n$ *after at most* cn^2 *steps is bounded below by a positive constant, given that* c *is large enough.*

Proof. Among the $N = cn^2$ steps the probability that the number X of steps that decrease the position equals k is $P(X = k) \leq \binom{N}{k} \cdot 2^{-n}$. The maximum of this term is reached for $k = N/2$ and by Stirling's formula it is bounded from above by $aN^{-1/2}$ for an appropriate a. Since $P(X = k)$ is decreasing left and right of the maximum, the probability of deviating by more than $n/2$ from $N/2$ is bounded above by the product $n/2 \cdot aN^{-1/2} = (na)/(2cn^2) = a/(2\sqrt{c})$. Conversely, the probability that X is at least $(1/2)N + (1/2)n$, meaning that we reach a state $m \geq n$, is at least $1 - a/(2\sqrt{n}) \geq \varepsilon > 0$ if c is large enough. □

As our second main technique, we show that it is sufficient to find a fitness-increasing mutation sequence of constant length to bound the success probability within $\mathcal{O}(n)$ steps by a positive constant from below.

Lemma 2. *Let* $v_1, \ldots, v_j$ *a sequence of vertices with* $j = \mathcal{O}(1)$*. If the sequence of mutations flipping* $v_1, \ldots, v_j$ *increases the fitness and is acceptable, then RLS increases the fitness within* $\mathcal{O}(n)$ *steps with positive constant probability.*

Proof. Denote by N all vertices connected to a vertex in $\{v_1, \ldots, v_j\}$. It holds that $|N| = \mathcal{O}(1)$. Since RLS does not accept fitness decreasing steps, we do not have to care about steps modifying vertices outside N. A mutation flipping v_i is accepted if $v_1, \ldots, v_{i-1}$ already flipped and none of the vertices in N flipped. For all i, v_i is the first vertex to flip in $N \cup \{v_i\}$ with probability at least $1/(|N| + 1)$. The probability for this to hold for all $i \in \{1, \ldots, j\}$ is at least $(|N| + 1)^{-j} \geq \varepsilon > 0$. The expected time for $\mathcal{O}(1)$ fixed bits to flip is $\mathcal{O}(n)$. □

3 Delimiting Blocks and Intersections

In this section, we will prove that bent borders can ensure a fitness gain within $\mathcal{O}(n^2)$ steps. Therefore we show that, whenever there are bent borders, there are also delimiting blocks or intersections. In this as well as in the following section we analyse RLS instead of the Metropolis algorithm, because it is likely to behave like RLS if p_T is small.

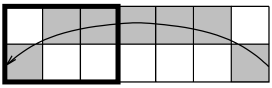
Lemma 3. *If there exists a bent border within a colouring of a torus, there also exists a delimiting block or an intersection.*

Proof. W. l. o. g. the border is not y-monotone and coloured 1. Then there must exist a path section starting with a non-zero y-component y_s and ending with a non-zero y-component $y_t = -y_s$, where the edges in between all have zero y-components. W. l. o. g. $y_s = 1$ and $y_t = -1$. The path section has the form

$$(x_s, +1), (x_h, 0), \dots, (x_h, 0), (-x_h, 0), \dots, (-x_h, 0), (x_t, -1).$$

The $(-x_h, 0)$ edges are usually absent. There are two cases:

1. $x_h = +1$, i. e. the path is bent to the right. Since all vertices left of the path are coloured 0, the vertices in the horizontal section form a delimiting block.
2. $x_h = -1$, i. e. the path is bent to the left. The vertices below the horizontal part of the border are coloured 0 and are a delimiting block candidate. Their upper, left, and right neighbours are border vertices and coloured 1, as is required by the definition of the delimiting block. However, the diagonal neighbours of the outermost 0-vertices are not necessarily coloured 1. If they are not, we have found an intersection.

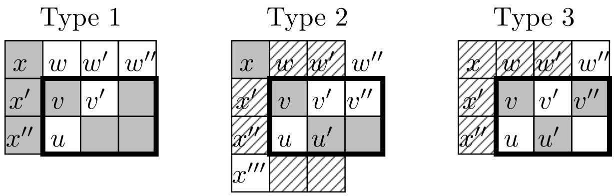
   In the image, we see a border (arrow) and a type 2 intersection (heavy frame). Type 1 and 3 are possible, if the length of the horizontal path section is 1.

Hence, a bent border contains at least one delimiting block or intersection. □

We will use Lemma 2 to show that fitness improvements are probable for all three types of intersections and for all situations, in which certain types of "failures" can occur during the random walk of the block length, which is analysed using Lemma 1.

Lemma 4. *If the current colouring contains an intersection, for RLS the probability of increasing the fitness within a phase of $\mathcal{O}(n)$ steps is bounded below by a positive constant.*

Proof. For all three types of intersections we will show that a sequence of not more than four mutations can increase the fitness by two, which, by Lemma 2 suffices to prove this lemma. In the following images, intersections are surrounded by heavy frames. The colouring of the neighbourhood results from considerations below.



Type 1 intersections. v' has four fixed 1-neighbours. Either v' can increase the fitness or the remaining three vertices w, w', and w'' above v' are coloured 0. Similarly, either v can increase the fitness or all three vertices x, x', and x'' left of v are coloured 1. If u and v' flip subsequently, the fitness increases.

Type 2 intersections. By definition of the intersection, the 1-vertex v has at least one fixed 1-neighbour. Either v can increase the fitness or it has three additional 1-neighbours. Since the 0-vertex and v-neighbour v' already has three fixed 1-neighbours, it can either increase the fitness or it may only have one additional 1-neighbour. Hence, at most one of the vertices w and w' may be coloured 1. Then at least two of the left neighbours x, x', and x'' of v must have colour 1. Since the situation is symmetrical, the same holds for the vertices below and left of u: Either u or u' can increase the fitness or at least two of the three vertices left of u are coloured 0. Hence, exactly one of the vertices x' and x'' are coloured 1 and the other is coloured 0, while x is coloured 1 and x''' is coloured 0. Now v can flip and u' gets a fifth 0-neighbour. Flipping u' increases the fitness.

Type 3 intersections. By definition of the intersection five of the neighbours of v' have a fixed colour. There are three cases:

1. v' has five or six 1-neighbours. v' can flip and increase the fitness.
2. All of the remaining neighbours of v' are coloured 0 and hence, v' has three 1-neighbours. Then the three 1-vertices of the intersection v, u', and v'' can flip subsequently and increase the fitness.
3. v' has exactly four 1-neighbours. We can place the remaining 1-neighbour of v' above the intersection at positions w, w' and w''. Let $w^* \in \{w, w', w''\}$ be this 1-neighbour of v'. Because of symmetry we can neglect the case that $w^* = w''$. So far, three of the neighbours of v are fixed to colour 0. If v has five 0-neighbours, it can increase the fitness. Otherwise, at least two of the three remaining v-neighbours x, x' and x'' left of v are coloured 1. If both x' and x'' are coloured 1, the vertices u and v' can increase the fitness if they flip subsequently (because v' has the 1-neighbour w^*). If at most one of the vertices x' and x'' is coloured 1, then v and u' can flip now. If $w^* = w'$, it can also flip. Finally, v'' has five 0-neighbours and can increase the fitness.

For all three types of intersections and all possible colourings of neighbours we found sequences of not more than four single-bit-mutations. □

Lemma 5. *If the current colouring contains a delimiting block, for RLS the probability of increasing the fitness within a phase of $\mathcal{O}(n^2)$ steps is bounded below by a positive constant.*

Proof. We have already seen that the outermost vertices of the delimiting block can always flip and decrease the block length by 1. Pessimistically, we assume that the outer neighbours of these vertices can also flip and enlarge the block length by 1. Unfortunately, there are some mutations which flip the neighbours of our block such that it loses its property of being surrounded by complementary vertices. This event F is referred to as a “failure.” We will show that the failure probability is not much larger than the probability of increasing the fitness ahead of time, which is referred to as event E. The phase terminates after $\mathcal{O}(n^2)$ steps or when the event $E \cup F$ occurs. We will show that $P(E|E \cup F) \geq \varepsilon > 0$.

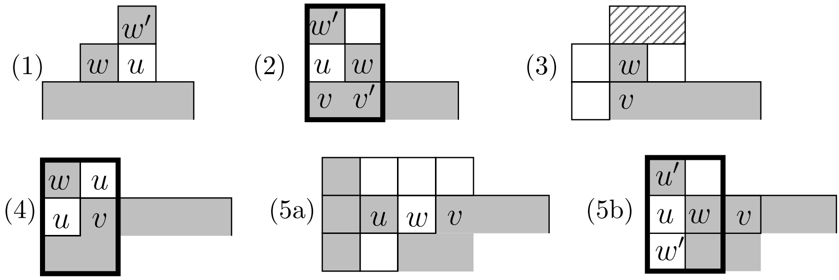
For now, let us take this inequality for granted and ignore all events from F. The block length performs a random walk where shrinking is at least as probable as growing, if $|B| \geq 2$. Since it cannot grow beyond $\sqrt{n}$, by Lemma 1 $\mathcal{O}(n)$ essential steps, i. e. steps changing the block length, suffice to reduce it to 1 with positive constant probability. The probability of an essential step is at least $1/n$. Hence with positive constant probability there is a sufficiently large number of essential steps among a phase of cn^2 steps if c is large enough. If $|B| = 1$, the probability of increasing the fitness by flipping the single vertex within $\mathcal{O}(n)$ steps is at least a positive constant. Multiplying all success probabilities still yields at least a positive constant.

The only thing left to do is to find a lower bound for $P(E|E \cup F)$. In order to do so, we apply the techniques used in the treatment of intersections. We will show that, whenever a failure is acceptable, there exists a sequence of mutations increasing the fitness. W. l. o. g. our delimiting block has colour 1 and is horizontal. Let B denote the set of vertices belonging to the block and let N denote the neighbours of B which are required to have colour 0.

There are two types of failures: First, a B-vertex can flip. However, this subdivides our block into two which is actually a shortening of B. Second, a vertex in N can flip. Denote the leftmost vertex of B by v and denote its right neighbour by v'. There are five different locations for an N-vertex: above one of the inner vertices of B (this covers actually $|B| - 4$ locations), above v, above v', left above v, and left of v (plus the locations at the right counterparts of v and v', respectively).

Before analysing the five cases we make a preliminary remark: By definition of the delimiting block, v has at least four 0-neighbours. If any of the three vertices below v has colour 0, v can flip and increase the fitness. Knowing that, we only have to cover colourings in which all lower neighbours of v are coloured 1. Furthermore, we can assume that the block length is at least 5. Otherwise, the mutation sequence flipping these vertices subsequently can increase the fitness.

Denote the row of the block by row 0, the row above by row 1, etc. (Apart from v and v' the scope of all names assigned to vertices in the case differentiation below is limited to the case they are defined in.)



1. A vertex w above an inner vertex (i. e. not v or v') of B flips. This step is only acceptable if w has at least one 1-neighbour w' in row 2. Now there is a 0-vertex u in row 1 which is a neighbour of both w and w'. Since v is located above an inner vertex of B, u also has three 1-neighbours from B in row 0. Altogether, u has five 1-neighbours and can increase the fitness.
2. The vertex w above v' flips. This case only differs from the first case, if the 1-neighbour w' of w in row 2 is in the column of v. Then we have a type 1 intersection (rotated by $90\,^\circ$) and we found a fitness-increasing mutation sequence sequence in the proof of Lemma 4.
3. The vertex w above v flips. By definition of the delimiting block, two neighbours of w are fixed to colour 1 and three are fixed to 0. In order for this step to be acceptable it is necessary that at least two of the three w-neighbours in row 2 (hatched) are coloured 1. In particular, one of the two vertices directly above and right above w must be coloured 1 which means that the 0-vertex u right of w has five 1-neighbours and can increase the fitness.
4. The vertex w left above v flips. Again, this is a rotated type 1 intersection.
5. The vertex w left of v flips. We have a failure iff the vertex u left of w or u' left above v is coloured 1. There are two cases.
   a) u is coloured 1. We analyse the situation *before* the flip of w. Including u, four of the neighbours of w are fixed to colour 1. Either the flip of w increases the fitness or its remaining neighbours are coloured 0. A similar argument holds for the remaining three neighbours of u. In this situation w and the vertex below u can flip subsequently and increase the fitness.
   b) u' is coloured 1. Furthermore, u is coloured 0 (otherwise we are in case 1). Independently of the colour of the the vertex w' below u, again, we have a type 1 intersection rotated by $90\,^\circ$.

Altogether we found fitness increasing mutation sequences that have a probability of at least $e > 0$ in $\mathcal{O}(n)$ steps for all situations in which failures are acceptable. Since the probability of the failure happening in the same time is at most 1, we have $P(E|E \cup F) \geq \frac{e}{e+1} > 0$, finishing the proof. □

4 Random Walk of Rings

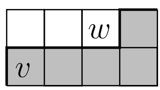
We have seen that bent borders vanish and we are left with rings delimited by monotone borders or stairs. Usually, the size of rings can perform a random walk and finally vanish, which is stated by the following lemma. However, there are special rings which cannot grow or shrink without decreasing the fitness. We call these rings and their borders "stable" and analyse them in Section 5.

Definition 4. *A monotone border p is called* stable *if all edges in p have exactly the same direction, i. e. all edges are horizontal, vertical or diagonal.*

Lemma 6. *If the current search point does not contain any stable borders and is non-optimal, then for RLS the probability of increasing the fitness within $\mathcal{O}(n^3)$ steps is bounded below by a positive constant.*

Proof. If we ever reach a colouring with bent borders, Lemmas 4 and 5 together with Lemma 3 state that we reach our goal of increasing the fitness within $\mathcal{O}(n^2)$ steps with positive constant probability. In the following we assume that the current search point is non-optimal and that there exists a monotone, but not stable border. We show that the optimum is reached within cn^3 steps with positive constant probability.

Let C be a monochromatic area delimited by monotone borders that are not stable. Without decreasing the fitness we may only flip vertices with at least four neighbours of complementary colour. Along monotone borders, these vertices are the corner vertices of steps as depicted below. Here, v and w can flip.



If the step has length at least 2, both corner vertices can change their colour, one of them increasing $|C|$ and one decreasing $|C|$. For every step of the stair we find one such pair of vertices. The number of possible $|C|$-decreasing and $|C|$-increasing steps is therefore identical and $|C|$ performs a random walk. If the step is part of a stable subsection of the border, these vertices cannot flip. Since there is at least one step which is not stable, the probability of an essential, i. e. $|C|$-changing step is at least $1/n$. By Lemma 1, cn^2 essential steps suffice for $|C|$ to shrink to 0 (which increases the fitness) with positive constant probability if c is large enough, which it is (again with positive constant probability) if we wait $c'n^3$ steps with c' a constant large enough. □

Taking into account that stairs typically have $\sqrt{n}$ steps we could save a factor of $\mathcal{O}(\sqrt{n})$. However, this would require us to show that not many of the steps are stable. The following theorem states that it is possible to transfer our previous results on RLS to the Metropolis algorithm.

Theorem 1. *Within $\mathcal{O}(n^4)$ steps, the Metropolis algorithm finds an optimum or a colouring without bent borders, if $p_T = \mathcal{O}(n^{-3})$ is sufficiently small.*

Proof. From Lemmas 4, 5, and 6 we know that RLS increases the fitness by at least 2 within $\mathcal{O}(n^3)$ steps with positive constant probability p_+, as long as the colouring does not entirely consist of stable rings. Also, by our choice of p_T

the probability that the Metropolis algorithm behaves exactly like RLS within $\mathcal{O}(n^3)$ steps is bounded below by a positive constant p_{RLS} which can be chosen arbitrarily close to 1. A phase ends after $\mathcal{O}(n^3)$ steps or when a fitness decreasing step is accepted. The expected fitness gain in each phase is bounded below by

$$p_{RLS} \cdot p_+ \cdot 2 + p_{RLS} \cdot (1 - p_+) \cdot 0 - (1 - p_{RLS}) \cdot 8,$$

since the fitness can decrease by at most 8 in one step. Choosing p_{RLS} large enough, both expectation value and the probability of observing a positive constant fitness gain are bounded below by a positive constant.

The fitness performs a random walk with a positive constant bias. Since negative steps are limited by 8 the expected number of phases until the fitness reaches its maximum of $4n$ is at most $\mathcal{O}(n)$. The probability of observing at most this number of phases is also a positive constant. Multiplying with the phase length $\mathcal{O}(n^3)$ yields a total number of $\mathcal{O}(n^4)$ steps.

These arguments hold as long as bent borders exist. If bent borders vanish, we accomplish the theorem ahead of time. □

5 Escape from Rings

In the preceding sections we did not make use of the property of the Metropolis algorithm of accepting search points with lesser fitness. Contrarily, we avoided it. Here we use this property for escaping from stable rings.

Theorem 2. *The expected optimisation time of the Metropolis algorithm with $p_T = \Theta(n^{-3})$ small enough is bounded above by $\mathcal{O}(n^{4.5})$.*

Proof. By Theorem 1 we know that the expected number of steps for reaching a colouring in which all borders are stable is bounded above by $\mathcal{O}(n^4)$. Then all mutations decrease the fitness. We start the phase by waiting for a step flipping a vertex near the border of a ring. The expected time until a step is accepted is hence p_T^{-1}. The probability that the vertex flipped in this step hits a border is at least $n^{-1/2}$ since the length of a monotone border is at least $\sqrt{n}$. If it does not hit a border it is likely to flip back before anything else happens. For now, let us assume that all bits flipping without hitting a border flip back before they cause any harm. Furthermore let us assume that no fitness decreasing steps are accepted within the phase described in the next paragraph.

Here we describe the case of a horizontal or vertical ring, but it should be clear that diagonal rings can be treated in exactly the same manner. The fitness decreasing mutation produced a delimiting block, or rather two: one of length 1 and one of length $\sqrt{n} - 1$. As we already know, the outer vertices of the blocks can flip lengthening or shortening it without decreasing the fitness. With probability 1/3 the single-vertex block does not flip back to its original state before growing to length 2. When the length of the block is at least 2, growing is at least as probable as shrinking and we can apply Lemma 1 again to obtain a positive constant lower bound on the probability of reaching length $\sqrt{n}$, and thereby changing the thickness of the original ring by 1, within $N = cn$ steps, c sufficiently large. However, now we may not reach length 1 or 0 *in between*, which seems very probable. We can calculate the success probability by applying the Ballot Theorem [1]. Given that there are $N/2 + \sqrt{n}/2$ steps lengthening the block and $N/2 - \sqrt{n}/2$ steps shortening it, the probability that the block never is shorter than 2 is

$$\frac{(N/2 + \sqrt{n}/2) - (N/2 - \sqrt{n}/2)}{(N/2 + \sqrt{n}/2) + (N/2 - \sqrt{n}/2)} = \frac{\sqrt{n}}{cn} = \frac{1}{c\sqrt{n}},$$

which is small, but not too small. In order to provide for cn essential steps with positive constant probability we need $c'n^2$ steps, c' large enough. The total length of the phase is dominated by the $\mathcal{O}(n^3)$ steps for waiting for the initial creation of the dent. Multiplying with the inverse of the success probability, we obtain $\mathcal{O}(n^{7/2})$ as a lower bound on the expected number of steps for changing the thickness of the ring by 1.

As we see, the thickness of the ring does now perform a random walk. In a colouring with i rings there exists, by the pigeon hole principle, at least one ring with thickness not greater than $\lfloor\sqrt{n}/i\rfloor$. Again, we apply Lemma 1 to retrieve a positive constant lower bound on the probability for this ring to vanish within $\mathcal{O}(\sqrt{n}^2/i^2)$ phases. This we call a meta-phase. Altogether, the expected length of a meta-phase is bounded above by $\mathcal{O}(\sqrt{n}^2/i^2) \cdot \mathcal{O}(n^{7/2}) = \mathcal{O}(n^{9/2}/i^2)$.

We still have to treat situations in which fitness decreasing mutations other than those described above are accepted. As long as vertices that decreased the fitness are isolated, they can always flip back which happens with probability $1 - \mathcal{O}(n^{-2})$ before the next step decreases the fitness. If it does not, there are four possibilities:

1. The flipping vertex never interferes with the process described above.
2. The flipping vertex collides with a delimiting block which is performing its random walk. The probability that a fitness decreasing vertex flips during the $\mathcal{O}(n^{5/2})$ steps of the phase in which there exists a delimiting block ($\mathcal{O}(\sqrt{n})$ trials with $\mathcal{O}(n^2)$ steps each) and that it is near the border is at most $\mathcal{O}(n^{5/2} \cdot n^{-3} \cdot n^{-1/2}) = \mathcal{O}(n^{-1})$. The probability that this happens at least once in a meta-phase (of at most $\mathcal{O}(n/i^2) = \mathcal{O}(n)$ phases) is at most $\mathcal{O}(1)$ which is a constant smaller than 1 if p_T is small enough.
3. The flipping vertex connects two rings. This is not a failure, but it is precisely what we are waiting for. With a similar argument we can conclude that with error probability $o(1)$ the width of the bridge is at most a constant. Then, a sequence of a constant number of acceptable mutations can increase the thickness of this bridge and the fitness. The expected time for this to happen is bounded above by $\mathcal{O}(n)$. Steps decreasing the width of the bridge also decrease the fitness and hence are not accepted with probability $1-\mathcal{O}(n^{-3/2})$ within $\mathcal{O}(n^{3/2})$ steps which is the expected time necessary for filling the entire area in between the former rings now connected by the bridge.
4. There are two or more competing bridges at a time, i. e. one 1-bridge connecting ring A and B and a 0-bridge bypassing one of the rings A and B. Given

that there is one bridge present, we have seen that the phase terminates within $\mathcal{O}(n^{3/2})$ steps is bounded below by a positive constant. The probability that within this time a second bridge (for which a fitness decreasing step is necessary) evolves, again, is bounded by $\mathcal{O}(n^{-3/2})$.

When a meta-phase is successful, it increases the fitness by at least $2\sqrt{n}$. Altogether, this happens with positive constant probability. Whenever a failure occurs in a meta-phase, we know from Theorem 1, that within $\mathcal{O}(n^4)$ steps, a situation with stable rings is restored, and that within this time the expected fitness gain is positive. Altogether the expected fitness gain of the meta-phase is bounded below by $\Omega(\sqrt{n})$. Summing up the lengths of the meta-phases for all values of i we retrieve an expected optimisation time of

$$\sum_{i=1}^{\sqrt{n}} \mathcal{O}(n^{9/2}/i^2) = \mathcal{O}(n^{4.5})$$

steps. □

6 Conclusion

The two-dimensional Ising model is an interesting model for the investigation of adaption processes with spatial interaction. We have proved that even a simple mutation-based search heuristic is able to leave local optima and converge to a global optimum within polynomial time by the means of random walk arguments on several levels. This is a surprising result since mutation-based algorithms were thought to be very slow even on the one-dimensional Ising model which can be seen as a subproblem frequently solved during the optimisation of the two-dimensional Ising model.

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