Parametric Reduced Order Modeling for Interconnect Analysis [*]

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Abstract— VLSI circuit models are subject to parameter variations due to temperature, geometry, process, and operating conditions. Parameter model order reduction is motivated by such practical problems. The purpose is to obtain a parametric reduced order model so that repeated reduction can be avoided. In this paper we propose two techniques: a nominal projection technique and an interpolation technique. The nominal projection technique is effective for small parameter perturbation by using a robust projection. The interpolation technique takes the advantage of simple matrix structure resulting from the PVL algorithm. A new moment matching concept in the discrete-time domain is also introduced, which is intended for a better performance in waveform matching and stability. Interconnect examples are used to test the effectiveness of the proposed methods.

I. Introduction

Gigahertz frequency operation is already a common practice in the current VLSI technology. Accurate interconnect modeling and analysis have gained increasing importance in state-of-the-art System-on-Chip (SoC) designs. The recent tutorial paper by Achar and Nakhla [1] presents a comprehensive review in this area.

Accurate interconnect models usually end up with high order. For fast analysis, model order reduction techniques have emerged as a valuable tool for such models. Also in many applications, models are likely parametric; for example, the geometric layout parameters, frequency dependent RLC values, and others. Parametric models facilitate synthesis and optimization. If a parametric model is large, it is favorable to have a reduced order model also parametric for efficient synthesis. Traditionally, parametric models are analyzed by repeated simulations and statistical experiments [3]. Interval analysis is another useful tool [13]. If a model is of high order, these methods are less efficient.

Although a number of model reduction techniques have been proposed in the literature (see a survey in [2]), almost all of them are numerical [10, 12, 7, 11]. They must be adapted to treat parametric models. Only a few research results reported in the literature deal with parametric models. Weile *et al.* treated two-parameter linear model reduction problem [15], where the parametric model matrices take the linear combination form as $p_1 M_1 + p_2 M_2$ with p_1 and p_2 being the parameters and M_1 and M_2 known matrices. The reason of choosing this special form is that the parameters are easily maintained after reducing the model order by a projective transformation. The projection is constructed from the Taylor expansion of the transfer function with respect to multiple parameters, from which moments are computed and matched. However, the number of moments in the multivariate expansion increases exponentially as the order increases, leading to an exponentially increasing computation for matching the high order moments. The idea of [15] is applied to multiple-line bus synthesis where the model parameters are wire spacing and wire width [6]. The work in [9] treats model involving parameters due to manufacturing variation. It applies the matrix perturbation theory of singular value decomposition and uses a parameter identification technique by assuming second order polynomial expressions for the variations of the dominant eigenvalue/eigenvectors and the congruence transformation. Because of the high complexity of the computation involved in this method, its applicability is very limited.

This paper is organized as follows. In Section II we briefly review the formulation of linear model order reduction and the projection method. Then we pose the general parametric model reduction problem and propose a nominal projection idea applicable to small range parameter perturbation. In Section III we introduce a new concept of moment matching in the discrete-time domain and argue that this new approach could possibly improve the robustness of nominal projection and the stability of the reduced model. An interpolation technique is introduced in Section IV, which takes the advantage of the simple matrix structure resulting from the PVL algorithm. Examples are presented in Section V. Finally, this paper is concluded in Section VI.

II. MODEL REDUCTION BY PROJECTION

We consider circuit models that are modeled by differential-algebraic equations

$$C\frac{dx}{dt} + Gx = Fu$$
$$y = Lx \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input (source) vector, and $y \in R^\ell$ is the output (measurement) vector. The transfer function of model (1) is

$$H(s) = L(Cs + G)^{-1}F. \tag{2}$$

The model order reduction problem is to find a reduced order model

$$\hat{C}\frac{d\xi}{dt} + \hat{G}\xi = \hat{F}u$$
$$y = \hat{L}\xi \tag{3}$$

where $\xi \in \mathbb{R}^q$ is the reduced state vector and $q \ll n$ is the reduced model order. The transfer function of the reduced model becomes

$$\hat{H}(s) = \hat{L}(\hat{C}s + \hat{G})^{-1}\hat{F}. \tag{4}$$

A practical requirement is that model (3) be a good approximation of the full order model (1) in the frequency domain.

Moment matching by projection is an efficient model order reduction technique [14]. Recent progress in numerical computation has made this approach widely accepted in practice [7].

Let W and V be two real matrices in $\mathbb{R}^{n \times q}$ so that $W^\mathrm{T}V$ is invertible. By restricting the state x in the subspace spanned by the columns of V, we can substitute x in (1) by $V\xi$ and pre-multiply the first equation by W^T to obtain a reduced order model (3) with

$$\hat{C} = W^\mathrm{T}CV, \ \ \hat{G} = W^\mathrm{T}GV, \ \ \hat{F} = W^\mathrm{T}F, \ \ \hat{L} = LV.$$

The two matrices W and V used in this reduction process are called projection matrices. They are constructed from standard algorithms for computing basis vectors of Krylov subspaces [7, 11]. If we choose $W = V$, then the transform is called the *congruence transform*.

Parametric linear time-invariant models can be described by

$$C(\beta)\frac{dx}{dt} + G(\beta)x = F(\beta)u$$
$$y = L(\beta)x \tag{5}$$

where $C(\beta)$, $G(\beta)$, $F(\beta)$, and $L(\beta)$ are model matrices depending on the parameter vector β containing a number of parameters. Ideally we would like to have a linear reduced order model that retains the same parameters. However, directly approaching this problem by symbolic linear algebra is clearly not feasible. In this paper we are interested in computationally feasible approaches to the parametric model reduction problem.

In many applications, the model parameters only perturb around some nominal values. For such cases, the parametric reduction can be formulated as a robust reduction problem. That is, we construct *nominal projections* from a set of nominal parameters and use them for reducing models with perturbed parameters. Since the foundation of projection-based reduction is subspace, we believe that the nominal subspace would possess certain degree of robustness if constructed appropriately. This idea will be tested in the experiment section.

The traditional projection algorithms are solely for matching moments in the frequency domain. Since here we are interested in a robust subspace in the time-domain, seeking a robust subspace construction method is one of the goals of this paper. In the next section a new moment matching concept is introduced for this purpose.

III. MOMENT MATCHING IN THE DISCRETE-TIME DOMAIN

There are many variants of Krylov subspace. One example of Krylov subspace is the one used in Complex Frequency Hopping (CFH) [4]. It is obtained by expanding the transfer function at some point in the complex plane

$$H(s) = L(Cs + G)^{-1}F = L\left[C(s - s_0) + (Cs_0 + G)\right]^{-1}F$$
$$= \sum_{i=0}^{\infty} LA^iB(s - s_0)^i \tag{6}$$

where $A = -(Cs_0 + G)^{-1}C$ and $B = (Cs_0 + G)^{-1}F$ and the matrix $(Cs_0 + G)$ is assumed to be invertible. If we construct a reduced order model which matches the leading moments (coefficients) of the above expansion, the reduced model approximates the original model at least in certain frequency range centering around $s_0 = j\omega$.

The Krylov subspaces used for matching the moments are generated by the triple [7, 11]

$$\left(L, \ (Cs_0 + G)^{-1}C, \ (Cs_0 + G)^{-1}F\right). \tag{7}$$

Standard Lanczos or Arnoldi algorithm can be used for this purpose.

Here we present a new approach to moment matching, which is from the time-domain perspective. If we discretize a continuous-time system using some discretization method, we obtain a discrete-time model. Then we can expand the transfer function in the z-domain and match the coefficients of those z^k terms, called z-moments.

For simplicity, we use the backward Euler formula to discretize the continuous-time model (1). The uniform time-step backward Euler formula is

$$\dot{x}_{k+1} = \frac{x_{k+1} - x_k}{h}$$

where $h > 0$ is the time-step length and $x_k = x(kh)$. After substitution, the continuous-time model (1) is discretized to

$$
\begin{aligned}
(\gamma C + G)x_{k+1} &= \gamma C x_k + F u_{k+1} \\
y_{k+1} &= L x_{k+1}
\end{aligned}
\tag{8}
$$

where $\gamma = 1/h$. Suppose $(\gamma C + G)$ is invertible, the state impulse response of (8) consists of the vectors

$$
\left\{ \Phi F, \ \Phi(\gamma C)\Phi F, \ [\Phi(\gamma C)]^2\Phi F, \ \cdots \right\}
$$

where $\Phi = (\gamma C + G)^{-1}$. These recursive vectors are clearly related to a Krylov subspace formed by the pair

$$
\left((\gamma C + G)^{-1}C, \ (\gamma C + G)^{-1}F \right).
\tag{9}
$$

The basis vectors of this Krylov subspace can be obtained by Arnoldi algorithm [11]. On the other hand, from the input-output point of view, one can use the triple

$$
\left(L, \ (\gamma C + G)^{-1}C, \ (\gamma C + G)^{-1}F \right)
\tag{10}
$$

to obtain a pair of bi-orthonormalized dual Krylov subspaces by Lanczos algorithm [7].

The Krylov subspace in (9) takes exactly the same form of rational Krylov subspace studied in Grimme's thesis [8]. However, Grimme came up with the same type of Krylov subspace from the shifted system of linear equations, from which the exact meaning of the real parameter γ is not clear. Furthermore, as we shall demonstrate below, moment matching in the z domain has the effect of waveform matching, which is also not observed in the Grimme's formulation. To avoid confusion we keep calling the Krylov subspace in (9) *rational Krylov subspace*.

Coincidentally, the rational Krylov subspace is also related to the moments by expanding the transfer function $H(s)$ at a positive real point γ by choosing $s_0 = \gamma$ in (6). Because of this simple connection, the rational Krylov subspace has already been used in many works, such as [7, 2], with good experimental results but without much justification. Grimme made some effort in his thesis ([8], Chapter 6), but did not reach a conclusive result.

The discrete-time moment matching can formally be described as follows. Let the columns of matrix V be the orthonormal basis vectors of the Krylov subspace in (9) obtained from the Arnoldi algorithm. Then the following identities hold:

$$
\left[(\gamma C + G)^{-1}C \right]^i (\gamma C + G)^{-1}F =
$$
$$
V \left[(\gamma \hat{C} + \hat{G})^{-1}\hat{C} \right]^i (\gamma \hat{C} + G)^{-1}\hat{F}
\tag{11}
$$

for $i = 0, 1, \cdots, q - 1$, where

$$
\hat{C} = V^{\mathrm{T}}CV, \quad \hat{G} = V^{\mathrm{T}}GV, \quad \hat{F} = V^{\mathrm{T}}F.
$$

This result is established in [8]. If we use V for projection and let

$$
\hat{Y}(z) = \hat{L} \left[(\gamma \hat{C} + \hat{G})z - \gamma \hat{C} \right]^{-1} \hat{F} z U(z)
$$

be the reduced order transfer function in the z-domain where $\hat{L} = LV$, then it follows that the leading q moments of $\hat{Y}(z)$ match those of $Y(z)$, which implies that

$$
\hat{y}_k = y_k, \quad \text{for } k = 0, 1, \cdots, q.
$$

This means that, starting from the same zero initial condition with the same input, the discrete-time responses of the full and reduced order models match at least in the first q steps. Note that matching a set of discretized points can be viewed as a constraint on the waveforms of the two models. Furthermore, due to the shifting property of discrete-time systems, matching the discretized points at the initial period may have a global effect, which implies global waveform matching in the time-domain. It would be interesting to derive an error bound in the time-domain based on this observation.

It is worth noting that a discretization by using the trapezoidal rule results in a Krylov subspaces same as in (9). This indicates from another angle that matching in a subspace is a much more general constraint than just matching several discrete-time points.

Remark 1 *The optimal choice of γ is not discussed here; in fact it is a further research topic. Since γ is the inverse of the time-step taken in discretization, it should be chosen so that the sampled state vectors have sufficient information for characterizing a model.*

IV. Interpolation method

The interpolation method is motivated by the simple matrix structure resulting from the PVL algorithm for model order reduction [7]. The input to the PVL algorithm is the matrix triple

$$
\left(L, \ (\gamma C + G)^{-1}C, \ (\gamma C + G)^{-1}F \right).
$$

We illustrate the interpolation principle by using a single-input-single-output (SISO) circuit model (MIMO cases can be treated similarly.) In the SISO case, we use the row vector ℓ^{T} in place of L and a column vector b in place of B.

The q step Lanczos algorithm (assuming no breakdowns) applied to the preceding triple (with $L = \ell^{\mathrm{T}}$ and $B = b$) yields a reduced order model in the state space

$$
\begin{aligned}
T_q \frac{d\xi}{dt} + (I - \gamma T_q)\xi &= e_1 u \\
y &= (\ell^{\mathrm{T}}b)e_1^{\mathrm{T}}\xi
\end{aligned}
\tag{12}
$$

with the transfer function

$$
\hat{H}(s) = (\ell^{\mathrm{T}}b)e_1^{\mathrm{T}} \left[T_q(s - \gamma) + I \right]^{-1} e_1,
\tag{13}
$$

where e_1 is the first column of the $q \times q$ identity matrix, $(\ell^{\mathrm{T}}b)$ is a scalar, and the matrix T_q is a $q \times q$ tridiagonal matrix. Thus, from the interpolation perspective, the free

parameters that determine a reduced order model include the scalar $(\ell^{\mathrm{T}} b)$ and the $3q-2$ possibly nonzero elements of the tridiagonal matrix T_q. Consequently, we can represent each reduced order model by a $(3q-1)$-dimensional vector.

The basic steps involved in the interpolation method are outlined here. Let $\mathcal{M}_i$, $i = 1, 2, \cdots, N$, be the N models sampled at the grid points of the parameters. Let $v_i \in \mathbb{R}^{3q-1}$ be the vector representing the models reduced from $\mathcal{M}_i$ by PVL. Reduced order models for new parameter values are obtained by interpolating the vectors v_i, $i = 1, 2, \cdots, N$.

There are many different ways to do interpolation. Certainly we should choose numerically simple but effective methods in that the computation should be many orders faster than running a whole reduction algorithm. For a single parameter, the Lagrange interpolation is a good candidate. Let p_i, $i = 1, 2, \cdots N$, be the grid points of a parameter p, which varies in the interval $[p_1, p_N]$. The basis polynomial for Lagrange interpolation is defined as (see [5], page 285)

$$\delta_i(p) = \prod_{\substack{j=1 \\ j \neq i}}^{N} (p - p_j) \bigg/ \prod_{\substack{j=1 \\ j \neq i}}^{N} (p_i - p_j). \qquad (14)$$

It is readily verified that $\delta_i(p_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta. Then for any $p \in [p_1, p_N]$, the model corresponding to p can be interpolated by

$$v(p) = \sum_{i=1}^{N} \delta_i(p) v_i.$$

For multiple parameters, there is a straightforward extension from the one-parameter interpolation by constructing multivariate basis polynomials from the direct product of single-parameter Lagrange basis functions as defined in (14). However, this approach is numerically not practical. Hence, in the multiple parameter case we use linear interpolation of the surrounding sample points, which is implemented as follows. Suppose we have K parameters, denoted by $\vec{p} = (p_1, p_2, \cdots, p_K)$. Each parameter is gridded separately and all sampled models are reduced by PVL. Then, given a new set of parameters, first the surrounding sample points are identified. Let $[p_k^\ell, p_k^r]$ be the smallest interval containing the kth parameter value p_k. Let $\mathcal{P}$ denote the set of parameter tuples $\{(p_1^t, \cdots, p_K^t) : t = \ell \text{ or } r\}$. Let $v_{(p_1^t, \cdots, p_K^t)}$ be the sampled model vector obtained at one of the surrounding sample points. Then the new reduced model at the parameter vector $\vec{p}$ can be written as

$$v(\vec{p}) = \sum_{(p_1^t, \cdots, p_K^t) \in \mathcal{P}} \left[\prod_{j=1}^{K} \delta \left([p_j^\ell, p_j^r], p_j^t, p_j \right) \right] v_{(p_1^t, \cdots, p_K^t)} \qquad (15)$$

where the function $\delta([a, b], p, x)$ is defined as one of the linear basis functions for interpolation over the interval $[a, b]$, i.e.

$$\delta([a, b], a, x) = \frac{x - b}{a - b} \quad \text{and} \quad \delta([a, b], b, x) = \frac{x - a}{b - a}.$$

There are two aspects of complexity involved in the interpolation method: one is the computation complexity and the other is the memory requirement. Depending on the operating frequency and the physical properties of interconnect, different reduced orders are needed to achieve acceptable analysis accuracy. Resistive interconnects can normally be analyzed by using very low order models with sufficient accuracy. However, inductive interconnects usually require higher order models to characterized the resonance effect at high frequencies; hence, the reduced model order should be chosen relatively higher. Obviously, the computation complexity of interpolation method depends on the order q, the number of parameters, and the number of sample models. For the reduction of each sample model, PVL is an extremely efficient algorithm (one LU factorization plus some matrix-vector multiplications).

For the purpose of interpolation, a number of reference models must be created first. The principle is similar to a look-up table. The reference models are created from the models sampled at the grid points of the parameters. Because of the possibility of exponentially increasing computation in the multi-parameter case, the number of parameters cannot be too large for applying the interpolation method. One can take the advantage of those insensitive parameters by using fewer number of grids for such parameters. For interpolation purpose, the reduced order models at the sampled parameter points must be stored in memory. The memory requirement is proportional to the product of the number of samples N and the reduced model order q.

Finally we mention that the stability of the interpolated model is a property related to the sample models. By continuity, the stability of all models for interpolation would likely imply that the interpolated model is stable as well.

V. Examples

The circuit shown in Fig. 1 is discretized from a one-dimensional interconnect or transmission line. Inductors are included in order to consider the inductive effect explicitly. For demonstration purpose, we assume that the model parameters characterizing different physical properties have been converted to the RLC values as parameters. The state space model is formulated by modified nodal analysis (MNA) with the nodal voltages and the currents passing the inductors as the state variables.

The nominal projection method is tested first. Note that higher order moments are needed for inductive interconnect analysis. In this test a 320th order model is reduced to 50th order with the voltage source as the input
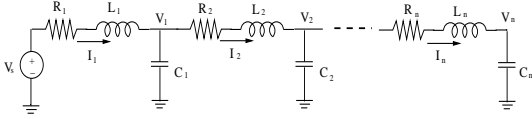
Fig. 1. An RLC line.

Fig. 3. Reduction of a perturbed model by nominal reduction. The dotted frequency plot indicates the nominal full order model for reference.

and V_1 (the voltage at node 1) as the output. The nominal RLC values are taken to be uniform with $R = 0.2\,\Omega$, $L = 1.0\,nH$, and $C = 0.5\,pF$. We choose a $\gamma = 10^9$ for discrete moment matching. Figure 2 shows the frequency responses of both full and reduced order models together with the error plot. The discrete moment matching yields a good approximation over a wide frequency band.

Then the RLC parameters are perturbed up to $\pm 50\%$ to test whether the reduced model maintains certain accuracy by using the nominal projection. Shown in Fig. 3 is the frequency response result and the error plot. The frequency response of the nominal full order model is also plotted (the dotted curve) to indicate the perturbation effect. Clearly, the frequency response of the reduced order model still captures the frequency response of the full order very well, but with a little sacrifice of the accuracy. An important observation from this test is that when a model is perturbed, the Krylov subspace associated the perturbed model actually is not perturbed much. Hence, a new model reduced by the nominal projection still captures the poles and zeros of the perturbed model. A theoretical justification of this experimental is under development.
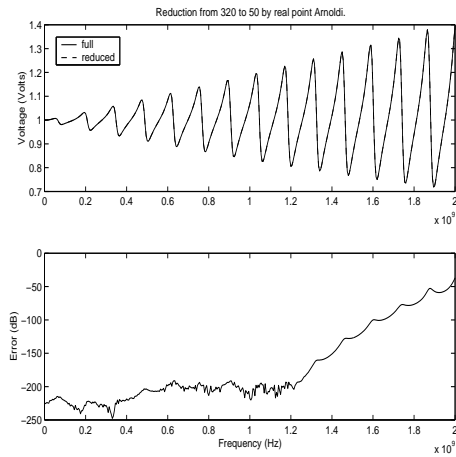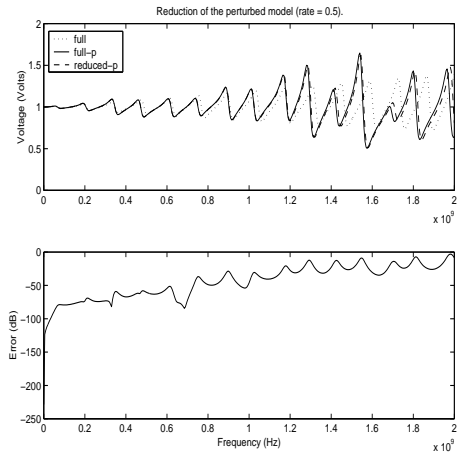
output. This model has only one parameter. The interval $[0.1, 1.0]$ is sampled by 11 equally spaced points and each of the 11 sampled models is reduced by PVL to 20th order. Then all other reduced order models are created by interpolation. Three test results are shown in Fig. 4 for $R_0 = 0.407,\ 0.694,\ 0.839\,\Omega$. It is clear that the resonance modes at high frequency are quite sensitive to the minor change of R_0 and the reduced models obtained by interpolation approximate the full-order model very well (as indicated by the error curves).
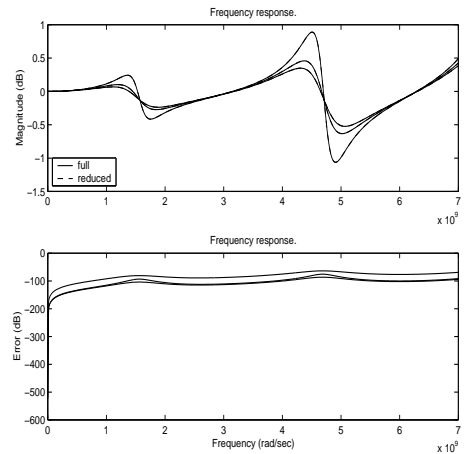

Fig. 2. Nominal reduction.

Fig. 4. Test results for one RLC line (three cases).

Next the same circuit in Fig. 1 is used to test the interpolation method. We assume that all R_i, L_i, and C_i take respectively the uniform values with $R_i = R_0$, $L_i = L_0$, and $C_i = C_0$, for all i and R_0, L_0, and C_0 are treated as three parameters of this model.

To study the sensitivity of R_0, we assume that R_0 varies in the interval $[0.1, 1.0]\Omega$, and $L_0 = 1.0\,nH$ and $C_0 = 0.1\,pF$ are fixed. The voltage V_1 remains as the

The interpolation method is also tested on another example with two coupled transmission lines as shown in Fig. 5. The input is $u = V_s$ and the voltage V_{11} is chosen as the output. The test case assumes that $R_{ij} = R_0$, $L_{ij} = L_0$, $C_{ij} = C_0$ for all i, j and $CC_i = CC_0$ for all i. Thus there are four parameters in this case. The interconnects are divided into 50 stages, resulting in a 200th order model. Each model is reduced to 20th order. For the plots in Fig. 6 we chose $R_0 \in [0.1, 0.2]\,\Omega$ with 4

grids, $L_0 \in [0.1, 0.5] \, nH$ with 3 grids, $C_0 \in [0.1, 0.5] \, pF$ with 3 grids, and $CC_0 \in [0.1, 0.2] \, pF$ with 3 grids. Thus in total we need to collect 108 sample models and reduce them by PVL. Show in Fig. 6 are the reduction results for the three randomly generated models and their reductions by linear interpolation over the surrounding points. We see that the frequency responses change drastically despite that the parameters only change mildly. Hence the resonance modes are very sensitive to the interconnect parameter variation. Regardless of the frequency response variation, the reduced models obtained by interpolation all approximate their full order models very well (see the error plots).
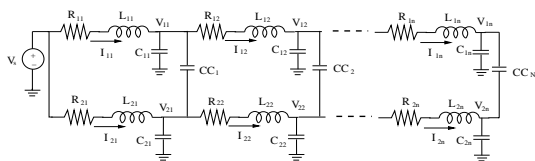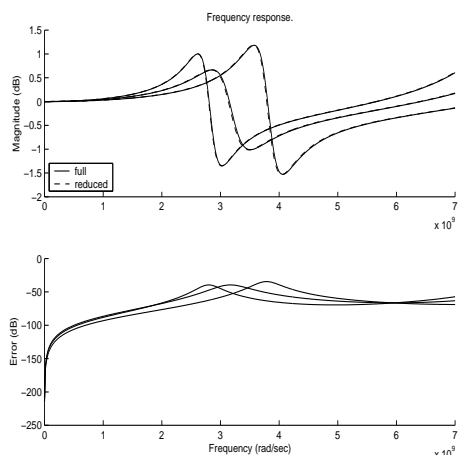

Fig. 5. Two coupled RLC lines.

Fig. 6. Test results for the coupled interconnect (three cases).

VI. Conclusion

Techniques for linear model order reduction have reached maturity. However, new problems still bring challenges to them. Parametric model order reduction is one of the practical problems, to which the traditional methods do not apply directly. Interconnect analysis is one of sources for such problems. Inductive effects of interconnect is being recognized as important for accurate delay measurement and design. This paper proposes two ideas for solving parametric model reduction under the assumption that the parameters have variations in a limited range. The application to interconnect analysis has been demonstrated by examples. Continuing research effort is needed for more general and effective methods.

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